2. (a) [2 marks] State the *Central Limit Theorem.*

(b) [10 marks] Let (R, S) be a pair of random variables with joint probability density function

$$f(r,s) = \begin{cases} \frac{1}{4}e^{-|s|}, & (r,s) \in [-1,1] \times \mathbb{R}, \\ 0, & \text{otherwise.} \end{cases}$$

Also consider independent identically distributed random variables (R_n, S_n), $n \geqslant 1$, with the same joint distribution as (R, S).

(i) Find the marginal probability density functions of R and S.

(ii) For any $s \in \mathbb{R}$, determine

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{\sqrt{n\,\mathrm{var}(S)}} \sum_{k=1}^{n} S_n \leqslant s\right).$$

(iii) For any $r, s \in \mathbb{R}$, show that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{\sqrt{n\,\mathrm{var}(R)}} \sum_{k=1}^{n} R_n \leqslant r,\ \frac{1}{\sqrt{n\,\mathrm{var}(S)}} \sum_{k=1}^{n} S_n \leqslant s\right) = \mathbb{P}(W \leqslant r, Z \leqslant s)$$

for a pair of random variables (W, Z) whose joint distribution you should determine.

(c) [13 marks] Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x - y, x + y)$. Let (R, S) be as in (b) and (X, Y) such that $(R, S) = T(X, Y)$.

(i) Derive the joint probability density function of (X, Y).

(ii) Find the marginal probability density functions of X and Y.

(iii) Find the correlation of X and Y.

2. Solution to Question 2 of A8 Probability 2022

(a) [2 marks] **B:** If V_n, $n \geq 1$, are iid with mean μ and variance $\sigma^2 \in (0, \infty)$, for all $v \in \mathbb{R}$

$$\mathbb{P}\Big(\frac{1}{\sqrt{n\,\sigma^2}} \sum_{k=1}^{n} (V_n - \mu) \leq v\Big) \to \mathbb{P}(Z \leq v) \qquad \text{as } n \to \infty, \text{ where } Z \sim N(0,1).$$

(b) **S/N** *(i)-(ii) are standard, (iii) is new, but straightforward due to independence.*

(i) [3 marks] Joint pdf factorises, so read off $f_R(r) = \frac{1}{2}$, $r \in [-1,1]$, $f_S(s) = \frac{1}{2}e^{-|s|}$, $s \in \mathbb{R}$.

(ii) [4 marks] Clearly S_n, $n \geq 1$, are iid with zero mean, by symmetry, and variance $\mathrm{var}(S) = \mathbb{E}[S^2] = 2\int_0^\infty s^2 \frac{1}{2} e^{-s} ds = 2 \in (0, \infty)$. Hence the CLT of (a) yields

$$\mathbb{P}\Big(\frac{1}{\sqrt{n\,\mathrm{var}(S)}} \sum_{k=1}^{n} S_n \leq s\Big) \to \mathbb{P}(Z \leq s), \qquad \text{for all } s \in \mathbb{R}, \text{ where } Z \sim N(0,1).$$

(iii) [3 marks] R_n, $n \geq 1$, are also iid with zero mean and $\mathrm{var}(R) = \frac{1}{3} \in (0, \infty)$.
As f factorises, R and S, and hence their partial sums are independent and by CLT

$$\begin{aligned}
&\mathbb{P}\Big(\frac{1}{\sqrt{n\,\mathrm{var}(R)}} \sum_{k=1}^{n} R_n \leq r, \frac{1}{\sqrt{n\,\mathrm{var}(S)}} \sum_{k=1}^{n} S_n \leq s\Big) \\
&= \mathbb{P}\Big(\frac{1}{\sqrt{n\,\mathrm{var}(R)}} \sum_{k=1}^{n} R_n \leq r\Big)\mathbb{P}\Big(\frac{1}{\sqrt{n\,\mathrm{var}(S)}} \sum_{k=1}^{n} S_n \leq s\Big) \\
&\to \mathbb{P}(W \leq r)\mathbb{P}(Z \leq s) = \mathbb{P}(W \leq r, Z \leq s),
\end{aligned}$$

where the joint distribution of (W, Z) is determined by independence and marginal distributions $W, Z \sim N(0,1)$.

(c) **S** *(i) is standard, so is (ii) but cases are tricky, (iii) risks inefficient attempts.*

(i) [4 marks] Linear transformation T, bijective with Jacobian determinant $J(x,y) = 1 \times 1 - (-1) \times 1 = 2$. By the transformation formula for pdfs, (X, Y) has joint pdf

$$f_{X,Y}(x,y) = |J(x,y)| f_{R,S}(T(x,y)) = \begin{cases} \frac{1}{2}e^{-|x+y|} & \text{if } |x-y| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) [6 marks] By symmetry X and Y are identically distributed. Also, for $|y| \geq \frac{1}{2}$,

$$\begin{aligned}
f_Y(y) = f_Y(|y|) &= \int_{|y|-1}^{|y|+1} \frac{1}{2} e^{-x-|y|} dx = \frac{1}{2}e^{-|y|}(e^{-|y|+1} - e^{-|y|-1}) \\
&= \frac{1}{2}(e - \frac{1}{e})e^{-2|y|}
\end{aligned}$$

and for $|y| \leq \frac{1}{2}$, we split into two integrals over $(|y|-1, -|y|)$ and $(-|y|, |y|+1)$:

$$\begin{aligned}
f_Y(y) = f_Y(|y|) &= \left(\frac{1}{2} - \frac{1}{2}e^{2|y|-1}\right) + \left(-\frac{1}{2}e^{-2|y|-1} + \frac{1}{2}\right) \\
&= 1 - \frac{1}{e}\cosh(2y).
\end{aligned}$$

(iii) [3 marks] $\mathrm{var}(Y) = \mathrm{var}(X) = \mathrm{var}(\frac{1}{2}S + \frac{1}{2}R) = \frac{1}{4}\mathrm{var}(S) + \frac{1}{4}\mathrm{var}(R) = \frac{7}{12}$ by independence. $\mathrm{cov}(X, Y) = \mathrm{cov}(\frac{1}{2}(S+R), \frac{1}{2}(S-R)) = \frac{1}{4}(\mathrm{var}(S) - \mathrm{var}(R)) = \frac{5}{12}$, so $\rho = \frac{5}{7}$.